PHYS 232 – Assignment #2

Due Friday, Feb. 16 @ 11:00

For all of these problems, to receive full credit, you must:

- Show all of your work.
- Explain your reasoning using words.
- Include large and neat diagrams when necessary.
- Present your solution in a neat a logical way that is easy to follow.

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1. Recall the simulations of suspended particles undergoing Brownian motion that were shown in class. In one of the simulations, the suspended particles were initially uniformly distributed and we watched how the distribution of particles evolved as they settled due to gravity. We saw that, after sufficient time, the number density $n(z)$ of the particles decayed exponentially as a function of the height z above the bottom of the container. In this problem, we attempt to analytically solve for this result. Specifically, we're trying to derive Eq. (16) in the Brownian motion lab manual which is reproduced here as Eq. (1):

$$n(z) = n_0 \exp\left[\frac{-\frac{4}{3}\pi a^3 (\rho - \rho_0)\, gz}{k_\mathrm{B}T}\right], \tag{1}$$

where n_0 is the number density at the bottom of the container ($z = 0$). The suspended particles are spheres of radius a and they are made of a material having density ρ. The solution in which the particles are suspended has a density $\rho_0 < \rho$. Finally, g is the gravitational acceleration and T is temperature in units of Kelvin. In this problem, we will treat the suspended particles as an ideal gas. Specifically, the particles do not interact with one another and they obey the ideal gas law $PV = Nk_\mathrm{B}T$. Here, P represents the *partial* pressure of the suspended particles. That is, it is the pressure that the particles in the suspension would exert if they alone occupied the entire volume occupied by the suspension.

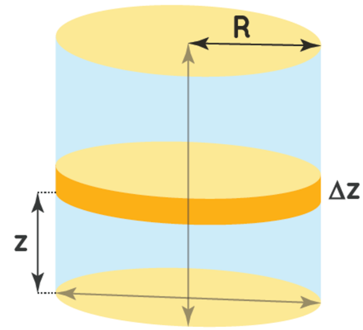

Figure 1: A cylinder of radius R. The volume of the yellow disk is $\Delta V = \pi R^2 \Delta z$.

(a) Consider the cylinder shown in Fig. 1. If the number density of particles a height z from the bottom of the cylinder is given by $n(z)$, show that the additional mass in the shaded volume element due to the presence of the particles is given by:

$$m = n(z)\pi R^2 \Delta z \left(\rho - \rho_0\right) \frac{4}{3}\pi a^3. \tag{2}$$

(b) Show that, in equilibrium, the following condition must be satisfied:

$$\frac{dP}{dz} = -n(z)\left(\rho - \rho_0\right)\frac{4}{3}\pi a^3 g. \tag{3}$$

Recall that P is the partial pressure of the suspended particles.

(c) Using the ideal gas law, show that:

$$\frac{dP}{dz} = k_\mathrm{B} T \frac{dn}{dz}. \tag{4}$$

(d) Finally, use Eqs. (3) and (4) to obtain the desired result:

$$n(z) = n_0 \exp\left[\frac{-\frac{4}{3}\pi a^3 \left(\rho - \rho_0\right) g z}{k_\mathrm{B} T}\right], \tag{5}$$

which reproduces the exponential behaviour observed in the simulation (after a sufficient number of iterations, or settling time). Note that a plot of $ln(n)$ versus z would result in a straight line whose slope can be used to determine k_B or, equivalently, $N_\mathrm{A} = R/k_\mathrm{B}$. This is precisely the measurement that is done on day 2 of the Brownian motion lab.

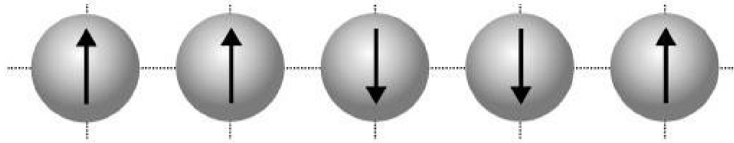

Figure 2: A 1-D array of spins. Each spin can either point up or down.

2. In this problem, the Binomial distribution will be used to study some properties of the 1-D Ising model. In this model, we consider a 1-D array of spins that can either point up or down. Figure 2 shows a series of five spins, but we'll ultimately imagine a large number of spins. The spin of a particle, like an electron or proton, acts as a small magnetic moment of strength $\mu_{\mathrm{B}} = e\hbar/(2m_{\mathrm{e}})$, a quantity called the Bohr magneton. Suppose that a spin that points up contributes $+\mu_{\mathrm{B}}$ to the system's magnetization M and that spins that point down contribute $-\mu_{\mathrm{B}}$.

(a) Use the binomial distribution to find the average magnetization $\langle M \rangle$. Assume that the probability that any individual spin points up is given by p. Does your result make sense for $p = 1/2$?

(b) Use the binomial distribution to find the standard deviation of the magnetization $\sigma_M^2 = \langle M^2 \rangle - \langle M \rangle^2$. Do you get the expected result for $p = 1/2$?

(c) When you take Statistical Mechanics (PHYS 403), you will see that one way to calculate the entropy S of a system in a given state is via:

$$S = k_{\mathrm{B}} \ln W \tag{6}$$

where W counts the number of configurations of that particular state. For example, there is only one way to arrange the spins such that they are all pointing up (or down). In this case, $W = 1$ and $S = 0$. Is this result consistent with your intuitive understanding of entropy?

(d) If, in a system of n spins, one is up and all others are down (or vice versa), there are $W = n$ arrangements. If we start with all of the spins down, we can select any one of the n spins to flip to the up position. In general, how many ways are there to have x spins up and $n - x$ spins down? For this part of the problem, it is okay to simply write down the answer.

(e) Your solution to part (d) should involve factorials. If we have a large number of spins, we'd have to evaluate factorials of large numbers which grow very quickly and are difficult to manage mathematically. Fortunately, *Stirling's approximation* can be used to re-express the factorial of a large number as:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{7}$$

where $e = 2.718\ldots$ is Euler's number. Try it for a modest number like $n = 10$. The approximation gets better and better as n increases.

Show that the number of arrangements of having half the spins up ($x = n/2$) and half the spins down can be expressed as:

$$W \approx \sqrt{\frac{2}{\pi n}}2^n. \tag{8}$$

(f) Finally, show that the probability of getting exactly have the spins up and half down when $p = 1/2$ can be approximated as:

$$P_{1/2} \approx \sqrt{\frac{2}{\pi n}}. \tag{9}$$

This last result is cute because, for $p = 1/2$, the average magnetization $\langle M\rangle = 0$ and the probability of getting exactly half spin up and half spin down is $P_{1/2} \propto n^{-1/2}$. Therefore, in the limit of large n (i.e. $n \to \infty$), we expect $\langle M\rangle = 0$, but the probability of getting half the spins up and half down goes to zero! The resolution is that there are many many arrangements with *nearly* half the spins up and half down for which we still satisfy $\langle M\rangle \approx 0$.